*i*th layer electrodynamics: A canonical approach

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(Received 22 March 1999; accepted for publication 27 April 1999)

Exact analytic expressions for the fields and the power dissipated in the *i*th layer of an *n*-layered structure are derived under steady-state and normal incidence via continuum electrodynamics. Via a transmission-line analog, we recursively propagate the surface wave impedance backward. We incorporate a canonical approach via three transfer functions that recursively propagates the field forward. The results apply exactly for an arbitrary number of layers, composed of arbitrary uniaxial materials, and having layers of arbitrary thicknesses. We consider examples of the electrodynamics of a superconducting thin film atop a dielectric and backed by a normal metal as a function of the thickness of the dielectric. © *1999 American Institute of Physics.* [S0021-8979(99)05215-9]

I. INTRODUCTION

Analytic solutions for the electrodynamics, including both the fields and the fractional power dissipation of the *i*th layer, of an *n*-layer stratified structure has been a long-sought-after effort. The optics literature has a formalism involving matrix methods which yields the net transmission and reflection (or dissipation).[1] The usual approach for systems involving only a few layers, however, is to bypass these matrix methods and instead resort to the familiar Maxwell's equations and explicitly solving or making assumptions about the boundary condition at each layer.[2–12] The fields are obtained and then integrated to obtain the power dissipation.[2,3] Early attempts[4–8] at the calculation of the transmission of fields and power across a superconducting film of thickness *d*, make the crude approximation that for $d \ll \delta_A$, where δ_A is the attenuation depth of a field in a film, the current density is constant throughout the film and the impedance at the incident surface (Z) is $Z = 1/\sigma d$. As we shall see, the effect of these two approximations implies a violation of the first law of thermodynamics. As in Refs. 4–8, other work[9] uses without proof this same Z for their value of *d*. These authors then claim to ''calculate *exactly* the transmission of a normally incident plane wave (emphasis added),''[9] but do not express the applicable range of *d* in terms of the impedance mismatch[10] or justify the application to their particular film thickness. Later work on a single film instead assumes that ''the spatial average of the electric field in the film is (given by the electric field at the center of the film),''[11] an approximation which breaks down at large thickness. Other works use a transmission line analog in order to compute the surface wave impedance exactly, and for all *d*.[2,12,13] The absence of an exact solution for the power dissipated in each region applicable for all film thicknesses for even a single film[14] speaks of the need for the work herein.

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II. SURFACE WAVE IMPEDANCE RECURSION RELATION

While retaining a transmission line analog in order to exactly obtain the surface wave impedance, we solve for a general boundary condition once and then obtain transfer functions which have application to each layer. This methodology provides exact analytical solutions to the fields in each layer of an *n*-layer stratified structure. As such, it can be used to compute the power being transmitted/reflected as well as the power dissipating in *each* layer (and not merely the *net* dissipation[1]). This feature is important to ascertain nonlinear behavior as a greater power density in the *i*th layer often correlates with nonlinearities in the *i*th layer's material parameters. These results are canonical in that they apply for an arbitrary number of layers being composed of arbitrary uniaxial materials (having planar isotropy) with arbitrary thicknesses. Our approach is also simpler than the matrix methods familiar to standard optics texts.[1] We begin with knowledge of the material parameters (e.g., γ_i and d_i, where d_i is the thickness of the *i*th layer and $\gamma_i = \alpha_i + j\beta_i$ is the familiar propagation constant for medium *i*) and by recalling that Ampere's law and Lenz's law mandate that the H and E fields, respectively, are continuous at each interface. Consequently, the ratio of these fields, called the wave impedance, will also be continuous everywhere. Using a transmission-line analog, we determine the (complex) wave impedance at the *i*th surface (Z_i) to be[13]

$$Z_i = \eta_i (Z_{i+1} M_i + \eta_i N_i) / (\eta_i M_i + Z_{i+1} N_i), \tag{1}$$

for $i = n-1, \ldots, 1$, where η_i is the intrinsic impedance of medium *i* and is given by $\eta_i = j\omega\mu_i/\gamma_i = \sqrt{\mu_i/\epsilon_{ci}}$, where ϵ_{ci} is the complex permittivity[15] [$\epsilon_{ci} \equiv \epsilon_i(1 - j\sigma_i/\epsilon_i\omega)$], $M_i = \cosh p_i + \cos q_i$, $N_i = \sinh p_i + j \sin q_i$, $p_i = 2\alpha_i d_i$, and $q_i = 2\beta_i d_i$. $Z_n \equiv \eta_n$. Equation (1) holds quite generally, and in practice, various limiting forms of γ and η can be expressed in terms of the conductivity, permeability, and permittivity for metals,[15] insulators,[15] dielectrics,[15] semiconductors, and superconductors.[13]

III. *H*-FIELD RECURSION RELATIONS

In general, the ith layer has a forward (A_i) and a refluent propagating complex H-field amplitude (in phasor notation). These fields combine to form a net field at the incident surface of the ith layer given by T_i. We can describe the magnitude of the forward field as it propagates some distance d within the ith layer by

$$|A_i(x=x'+d)|=|A_i(x=x')|e^{-d\alpha_i} \tag{2a}$$

for $i=1,...,n$. Continuity of the fields mandates that,[2]

$$\left|\frac{A_i(x_i^+)}{A_{i-1}(x_i^-)}\right|=\left|\frac{1+Z_i/\eta_i}{1+Z_i/\eta_{i-1}}\right|,$$

for $i=1,...,n$ where the + (−) superscript denotes the infinitesimal displacement to the right (left) of the boundary separating the $i-1$th layer from the ith layer located at x_i (assuming the incident wave originates on the left). Combining this equation with Eq. (2a), we obtain

$$\left|\frac{A_i}{A_{i-1}}\right|=e^{-d_{i-1}\alpha_{i-1}}\left|\frac{1+Z_i/\eta_i}{1+Z_i/\eta_{i-1}}\right|, \tag{2b}$$

for $i=2,...,n$ where the omission of the spatial coordinate of the forward field amplitude signifies that the forward field is to be evaluated at the incident face (i.e., the left-most end if the field originates on the left) of the layer denoted by the subscript. Similarly, we modify the transfer function relating the incident field to the tangential field[2] by Eq. (2a), to obtain

$$\left|\frac{T_i}{A_{i-1}}\right|=e^{-d_{i-1}\alpha_{i-1}}\left|\frac{2\eta_{i-1}}{\eta_{i-1}+Z_i}\right|, \tag{2c}$$

for $i=2,...,n$. From Eq. (2) we can solve for the fields over all space in each medium.

IV. POWER DISSIPATION IN THE *i*th LAYER

Next, we create a Gaussian pill box whose axis is aligned with the incident wave vector and whose end caps are at both surfaces of the ith layer. Integrating the Poynting vector over its surface and dividing by the incident power (P_{in}), we find

$$\frac{P_i}{P_{\text{in}}}=\frac{1}{\eta_0}\left(\frac{|T_i|^2}{|A_0|^2}\mathscr{R}(Z_i)-\frac{|T_{i+1}|^2}{|A_0|^2}\mathscr{R}(Z_{i+1})\right), \tag{3}$$

for $i=1,...,n-1$, where P_{in} is given by $P_{\text{in}}=\eta_0|A_0|^2/2$ and $\mathscr{R}$ is the real operator. [From Eq. (3) we see that if one models the field in the film as constant throughout the film and takes $\mathscr{R}(Z_{i+1})>\mathscr{R}(Z_i)$, as is done in Refs. 4–9, then $|T_i|=|T_{i+1}|$ and negative energy would ostensibly dissipate in the ith layer.] In a straight-forward application of Eqs. (2) and (3) we find the fractional power dissipating in the ith layer:

$$\frac{P_i}{P_{\text{in}}}=\frac{4}{\eta_0}e^{-\Sigma_{k=1}^{i-1}p_k}\prod_{j=1}^{i-1}\left|\frac{1+Z_j/\eta_j}{1+Z_j/\eta_{j-1}}\right|^2\frac{1}{|1+Z_i/\eta_{i-1}|^2}\times\left[\mathscr{R}(Z_i)-e^{-p_i}\frac{|1+Z_i/\eta_i|^2}{|1+Z_{i+1}/\eta_i|^2}\mathscr{R}(Z_{i+1})\right], \tag{4a}$$

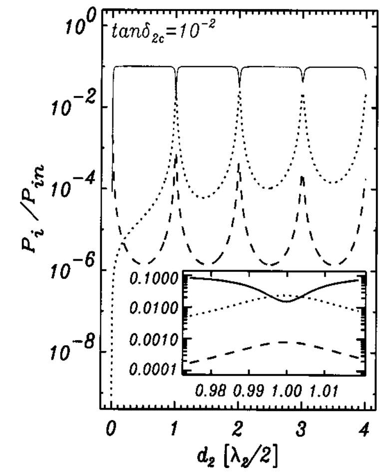

FIG. 1. Normalized dissipation in each of three layers of a metal–dielectric–metal composite as a function of the thickness of the dielectric. Inset: detail about a thickness of one-half wavelength of the dielectric.

for $i=1,...,n-1$. From $P_n/P_{\text{in}}=|T_n/A_0|^2\mathscr{R}(\eta_n)/\eta_0$, we find the fractional power in the last or nth layer to be

$$\frac{P_n}{P_{\text{in}}}=\frac{4}{\eta_0}\frac{e^{-\Sigma_{k=1}^{n-1}p_k}}{\left|1+\frac{\eta_n}{\eta_{n-1}}\right|^2}\prod_{j=1}^{n-1}\left|\frac{1+Z_j/\eta_j}{1+Z_j/\eta_{j-1}}\right|^2\mathscr{R}(\eta_n). \tag{4b}$$

Finally, via the relationship[2] $P_t/P_{\text{in}}=4\eta_0\mathscr{R}(Z_1)/|\eta_0+Z_1|^2$, which can also be evaluated by summing over Eq. (4) (i.e., $\Sigma P_i/P_{\text{in}}$), we have a self-consistent method for confirming the accuracy of Eq. (4).

V. EXAMPLE: SUPERCONDUCTING THIN-FILM–DIELECTRIC SUBSTRATE–BULK METAL

Because of the voluminous work on films which are superconducting, we consider an experimentally common example of a superconducting film on a dielectric substrate, followed by a normal metal (sdm). Although transmission-line analogs have been previously applied to superconducting film–dielectric systems,[12,16] losses in the dielectric have been ignored. To obtain numerical results, we consider this system to be driven at 10 GHz and take the relative permeability and permittivity to be unity for all mediums.

Although analytic expressions for the fields and the power dissipation of a superconducting film atop a bulk metal has been recently obtained,[13] this has not been done for the sdm structure. Regarding this basic structure, Ceremuga, Barton, and Miranda[17] claim, ''It is obvious that the addition of an intermediate layer to a superconducting structure can only have a significant effect on the propagation properties when thickness (sic) of the buffer approaches one quarter of the characteristic wavelength.'' However, we note otherwise. Figure 1 illustrates the power dissipated in each medium (film: solid, dielectric: dotted, and bulk metal: dashed), as a function of the thickness of the dielectric, while the film is in the *normal* state. The conductivity of both the metallic film and the bulk metal is taken to be 10^6 Mhos/m

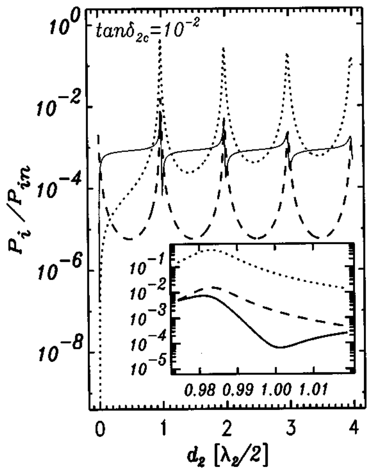

FIG. 2. Normalized dissipation of each of three layer superconductor–dielectric–metal composites as a function of the thickness of the dielectric. Inset: detail about a thickness of one-half wavelength of the dielectric.

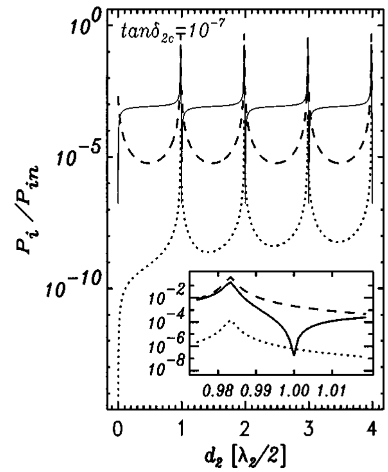

FIG. 3. Normalized dissipation of each of three layer superconductor–dielectric–metal composites as a function of the thickness of the dielectric. Inset: detail about a thickness of one-half wavelength of the dielectric.

($\delta \sim 5.0\ \mu$m), while the loss tangent of the dielectric is 10^{-2} (using the material descriptors of Ref. 13, this implies $\lambda_2 \approx 3$ cm, $\delta_{A2} \approx 0.95$ m, and $\delta_{P2} = \lambda_2/2\pi \approx 4.8$ mm). The thickness of the film is fixed at 0.1 μm. For a vanishing dielectric thickness, the data of Fig. 1 yield $P_{\text{film}}/P_{\text{in}} \approx 8.26 \times 10^{-5}$ and $P_{\text{metal}}/P_{\text{in}} \approx 2.02 \times 10^{-3}$, in precise agreement with the results from analytical expressions for a film on a metallic substrate.[2,13,18] The self-consistency check discussed in Sec. IV reveals agreement to over 30 significant figures for each of the >3500 data points considered in Figs. 1–3. The results of Fig. 1 reveal that if the dielectric were removed, the dissipation in the bulk metal would be $P_t/P_{\text{in}} \approx 2 \times 10^{-3}$, which, except for the thicknesses near the resonant dielectric thickness, is about *2 orders of magnitude less than the total power dissipation that occurs in the film alone when the dielectric is present and the system is far away from resonance*. At integer multiples of 1/2 a dielectric wavelength, we see resonant thicknesses indicating standing waves in the dielectric. The inset reveals that even for our modest loss tangent of 10^{-2}, the dissipation in the dielectric can exceed that of the film and the metal for d_2 near $\lambda_2/2$. We also see that at the higher-order resonances, the dissipation in the film increases and the dissipation in the bulk metal decreases.

From Fig. 1 we also see a qualitatively similar dissipation profile in the dielectric and the bulk metal for $d_2 > \lambda_2/4$. We can explain this by appealing to Eq. (2c) and by recognizing that there are no reflections in the metal.

Next, we consider the same structure, but take the superconductor to be in the *superconducting* state. To maintain a pedagogical continuity to the film of Fig. 1, we take the complex penetration depth of the superconductor to be given by $\tilde{\lambda} \approx 5\ \mu\text{m} - i\,10\ \text{nm}$ [$\delta_A \approx 5.0\ \mu$m (i.e., the same as in Fig. 1), and $\delta_P \approx 2.5$ mm, which is also equivalent[13] to specifying $\mathscr{R}(\tilde{\sigma}) \approx 2$ kMho/m and $\mathscr{I}(\tilde{\sigma}) \approx -0.5$ MMho/m, where $\mathscr{I}$ is the imaginary operator]. The result of this work is shown in Fig. 2, where—as compared with Fig. 1—an additional structure surrounding the resonant thickness is seen. The only difference between the structures of Figs. 1 and 2 is that δ_{P1} is changed from $\sim 5.0\ \mu$m to ~ 2.5 mm. This effect is acutely manifested in the wave impedance at each surface, as will be discussed in a subsequent publication. We note that the inset of Fig. 2 reveals that the minima in the dissipation of the film are not coincident with the maxima of the dielectric and the metal, as it is in the inset of Fig. 1, and that the dissipation in the superconductor sharply increases then decreases as the dielectric thickness increases through a resonance. The change of *a mere few percent* in the thicknesses of the dielectric, changes the dissipation in the superconductor by over *two orders of magnitude*. Curiously, the dissipation in the dielectric has increased for nearly all thickness with this change in the phase length scale in the film and for $d_2 \approx \lambda_2/2$, nearly approaches the incident power. For the range of d_2's considered in Fig. 2, there is a sizeable range of thicknesses where the total power dissipation is dominated by the dielectric's.

Finally, Fig. 3, which represents the same structures as Fig. 2 except $\tan \delta_{2c}$ is changed from 10^{-2} to 10^{-7} (corresponding to a change *from* $\delta_{A2} \approx 0.95$ m and $\delta_{P2} = \lambda_2/2\pi \approx 4.8$ mm *to* $\delta_{A2} \approx 95$ km and $\delta_{P2} = \lambda_2/2\pi \approx 4.8$ mm), reveals how changing the thickness of the dielectric in and out of resonance serves to couple and decouple the film to the metal. As with Fig. 2, for $d_2 = 0$ Fig. 3 reveals that $P_1/P_{\text{in}} \approx 1.70 \times 10^{-7}$ and $P_2/P_{\text{in}} \approx 2.06 \times 10^{-3}$ in agreement with other work.[13] However, these are also the values of P_1/P_{in} and P_2/P_{in} at $d_2 = n\lambda_2/2$ in Fig. 3. *Thus, the low-loss dielectric couples the film and the metal at the resonant thicknesses*. This behavior is *opposite* to that predicted by Ref. 17 as noted earlier. Making the dielectric more lossy inhibits this ability, especially for large n. From the inset of Fig. 3 we see that a change of *a mere few percent* in the thickness of the dielectric changes the dissipation in the superconductor by over *six orders of magnitude* and the total dissipation by about *two orders of magnitude*. At resonance, we see that even though the dissipation in both the superconducting film and the bulk metal return to their values for the case where there is no dielectric, they do so only after exhibiting orders of magnitude greater dissipation when the dielectric thickness is less than a couple percent smaller than the resonant value (see the detail of Fig. 3). From the inset of Fig. 3 we also note a power dissipation maxima of all three materialscoinciding at about a couple percent less than $\lambda_2/2$—behavior that is very different than that of Fig. 1.

VI. CONCLUSION

Having developed a canonical formalism for addressing exact electronic transport behavior in stratified media composed of arbitrary materials, with arbitrary thickness, and with an arbitrary number of layers, we find this approach to be fruitful both pedagogically and computationally, where the cumbersome operations of matrix methods are avoided. Some of the pedagogical fruit corrects notions about standing waves in dielectrics and putative approximations about the surface wave impedance and the current density distribution in the very thin-film limit. The generalizability of this formalism permits ready application to quantitative material analysis—where knowledge of P_i can assess nonlinear effects—as well as application to a diverse set of problems, be they transmission $(P_n/P_{\rm in})$, dissipation $(\Sigma P_i/P_{\rm in})$, or reflection $(1-P_t/P_{\rm in})$. Although we have only considered the experimentally important case of normal incidence—while optical matrix methods admit oblique incidence—[1] we provide the expressions for the dissipation in the *i*th layer, and not merely the net dissipation.[1]

ACKNOWLEDGMENT

The author gratefully acknowledges Emily J. Beeli for a generous grant-in-aid.

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